4–2. Interpolation and Extrapolation

Suppose we measure the volumes and corresponding radii of a number of spheres and plot the results (volume against radius). From the measurements, we are sure of the positions of a number of points on this plot, one for each sphere. If we now draw a smooth curve through these points, we obtain a curve from which we can find the volume of a sphere for any radius — not only for the values of the radii we have measured. The process of finding from this plot new values located between the measured ones is called *interpolation.* Such a process is meaningful and useful when there are good reasons to believe that the curve is valid for values between the measured ones. Then one gets information which is not immediately available from the measurements.

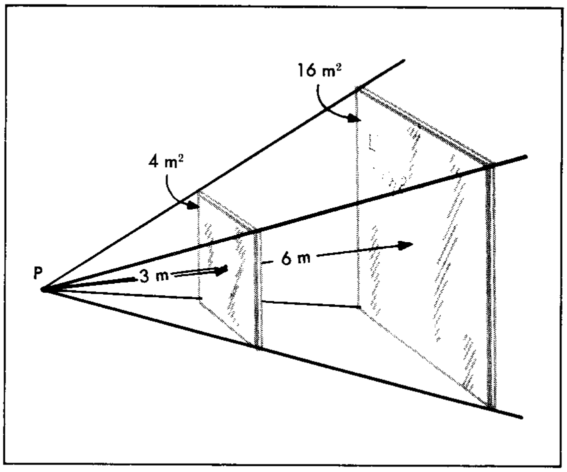
In the example of the relation between the volumes and radii of spheres, we know from the equation $V = \frac{4\pi}{3} R^3$ that the volume changes smoothly with the radius. So a smooth curve through a few computed or measured points is reasonable. When no formula is known, however, we depend only on experimental measurements. Then drawing a smooth curve expresses our belief that things change smoothly in nature. Interpolation always carries with it some element of risk. Even if things do change smoothly, we must get experimental values quite close together if we want to know how the graph goes in any region where it curves sharply. Interpolation is of no use at all for graphs of functions which cannot be represented by smooth curves.

Extrapolation, carrying the plot out beyond the range of the data, is even more risky. Error can arise here more easily, but so can discovery. For example, the problems encountered by aircraft in breaking the sound barrier were foreseen by extrapolation of equations which describe the exact behavior of aircraft at speeds well below that of sound. Extrapolation from the behavior of gases at normal temperatures leads to the idea of a lowest possible temperature, absolute zero, but about objects traveling close to the speed of light, extrapolation from ordinary experience leads to nonsense.

In our examples of the volumes of a series of spheres and the areas of squares, extrapolation would be quite as safe as interpolation, for we know the equations to hold according to the geometry of Euclid for spheres or squares of any size, however large. But the physicist has to admit that he has no certain test for the validity of Euclid's geometry beyond the distances to the galaxies. Indeed, theoretical physicists have invented proposals to change the laws of Euclid whenever enormous distances are involved. From the point of view of physics, the geometry of space is subject to experiment. Euclid's geometry may not be an accurate description of our measurements if the shapes we study range in size over many orders of magnitude. Naturally, we shall not change our description unless it gets us into trouble. In this course the geometry of Euclid will serve us well.

4–3. The Inverse-Square Relation

Look at a row of street lights that stretch away from you in the distance. The lamps themselves are all the same — that is, each gives off the same amount of light each second — but the closer each one is to you the more intense it appears. If the light spreads out equally in all directions (which is nearly true for a street lamp and a star and many other sources), it can be pictured as shown in Fig. 4–6. Here we consider just a portion of the light moving out through a sort of "pyramid" from the point P. As the distance



4–6. The inverse-square relation. Light from a point (P) radiates in all directions. Since the light spreads out to cover four times the area at twice the distance, it follows that it can be only $\frac{1}{4}$ as intense. Thus, when the distance is doubled, the intensity decreases to $\frac{1}{4}$, or the intensity is inversely proportional to the square of the distance.

from the source increases, the light is spread over a greater area and the light appears less intense. This suggests that the intensity of the light is inversely proportional to the area it falls on.

$$I \propto \frac{1}{A},$$

where I represents the intensity and A the area. For the moment let us assume that this relation holds for light. Later you will study light intensities experimentally.

Each of the sides of the squares in Fig. 4–6 is proportional to its distance from P. Therefore, the area of each square is proportional to the square of that distance. If we call the distance d, this can be expressed as

$$A \propto d^2.$$

Combining this relation with $I \propto \frac{1}{A}$ we find that

$$I \propto \frac{1}{d^2}.$$

This is the inverse-square relation, which for light says that the intensity is inversely proportional to the square of the distance from the source.

In detail you can see that $I \propto \frac{1}{d^2}$ by remembering that

$$I \propto \frac{1}{A}$$

means $$\frac{I'}{I} = \frac{A}{A'} \qquad (1)$$

and that $$A \propto d^2$$

means $$\frac{A}{A'} = \frac{d^2}{(d')^2}. \qquad (2)$$

So, combining (1) and (2) gives

$$\frac{I'}{I} = \frac{d^2}{(d')^2}. \qquad (3)$$

This is the same as

$$I \propto \frac{1}{d^2}.$$

Note that relation (3) holds for a single source at distances d' and d. It also holds for two identical sources, one at distance d' and the other at distance d. For example, suppose we have two street lamps, which we call 1 and 2, at different distances d_1 and d_2 from a white wall which they illuminate. Then, their intensities at the wall are in the ratio

$$\frac{I_1}{I_2} = \frac{d_2^2}{d_1^2}.$$

This relation enables us to estimate the distance of one lamp if we have another equal lamp at a known distance. For example, suppose we find that a lamp 10 meters (d_1) away gives an intensity that is 16 times that of an identical lamp at some unknown distance d_2. (Photoelectric cells, camera light meters, and photographic plates can give accurate measures of relative intensity. So can the eye with the aid of a special screen on which to make comparisons.) How can we find d_2? We know that I_1/I_2 is 16 and we know that d_1 is 10 meters.

$$16 = \frac{I_1}{I_2} = \frac{d_2^2}{(10 \text{ meters})^2}.$$

Solving for d_2 we get

$$d_2 = \sqrt{16 \times (10 \text{ meters})^2} = 4 \times 10 \text{ meters}$$
$$= 40 \text{ meters}.$$

This is just the method which gives us our knowledge of the distance of far-off stars, whose distance from us is too great to be measured by the geometric methods using the diameter of the earth's orbit as a base line. The measurement is made by comparing the intensity of the faint image of a faraway star on a photographic plate with the intensity of a near-by star which appears to give off the same amount of light. The measurement is at best a rough one, because we do not expect that the two stars are really equally strong sources of light. But in this rough way we can go far beyond the possibilities of triangulation methods and at least determine the order of magnitude of the distance to far-off stars.

We can see how the inverse-square relation works by measuring the distance of a near-by star, using the inverse-square relation and comparing our result with the distance measured geometrically. There is a good star for this purpose, α Centauri A. Judging from its color and calculated mass, this star is very similar to the sun. But the intensity of illumination here at the earth is 10^{11} times greater from the sun than from α Centauri A. From the inverse-square relation